Frequency Estimation of Sinusoidal Signals in Alpha-Stable Noise Using Subspace Techniques*

Mustafa A. Altınkaya, Hakan Deliç†, Bülent Sankur, Emin Anarım
Signal and Image Processing Laboratory
Department of Electrical and Electronics Engineering
Boğaziçi University
Bebek 80815 Istanbul, Turkey
{altink, delic, sankur, anarim}@busim.ee.boun.edu.tr

Abstract

In the frequency estimation of sinusoidal signals observed in impulsive noise environments, techniques based on Gaussian noise assumption are unsuccessful. One possible way to find better estimates is to model the noise as an alpha-stable process and to use the fractional lower order statistics of data to estimate the signal parameters. In this work noise and signal subspace methods, namely MUSIC and Principal Component-Bartlett, are applied to fractional lower order statistics of sinusoids embedded in alpha-stable noise. The simulation results show that techniques based on lower order statistics are superior to their second order statistics-based counterparts, especially when the noise exhibits a strong impulsive attitude.

1. Introduction

Most of the work on the frequency estimation problem assumes that the additive noise has Gaussian distribution. This is partly because of the nice properties of the Gaussian model which allows for simplification of the theoretical work and decreases the computational complexity in signal parameter estimation. As long as the noise distribution can fit approximately to a Gaussian model, in particular for the tails of the distribution, one can obtain good estimators with the Gaussian noise assumption. But if the noise process belongs to a non-Gaussian, especially a heavily-tailed, distribution class or when the noise is of impulsive nature, parameter estimators which are based on Gaussian noise assumption break down.

Impulsive noise processes can be modeled using stable distributions. If a signal can be thought of as the sum of a large number of independent and identically distributed random variables, the limiting distribution will be in the class of stable distributions according to Generalized Central Limit Theorem [5], and stable distributions cover Gaussian distribution in the limit.

If the additive noise has a heavily-tailed distribution which is successfully modeled by alpha-stable distributions, the performance of covariation-based frequency estimators is better than that of the traditional estimators which are based on second order statistics.

In this work subspace-based estimation methods using covariations are considered. In Section 2, the $S\alpha S$ distributions are briefly discussed. In Section 3, the application of fractional lower order moments (FLOM) to frequency estimation problem is presented. Section 4 covers the results of the simulation experiments. Finally conclusions are in Section 5.

2. $S\alpha S$ Distributions

An important sub-class of stable distributions are symmetric alpha-stable ($S\alpha S$) distributions. The characteristic function of $S\alpha S$ variables is given by:

$$\phi(\omega) = \exp\{j\delta\omega - \gamma|\omega|^{\alpha}\} \tag{1}$$

where α is the characteristic exponent ($0 < \alpha \leq 2$), δ is the location parameter ($-\infty < \delta < \infty$) and γ is the dispersion ($\gamma > 0$). Without losing generality we may take the location parameter $\delta = 0$ as in the zero mean Gaussian noise assumption case. This assumption will

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†On leave from the Department of Electrical and Computer Engineering, University of Southwestern Louisiana, Lafayette, LA 70504-3890, USA.

lead to the characteristic function:

$$\phi(\omega) = \exp\{-\gamma|\omega|^{\alpha}\}\,. \tag{2}$$

For $S\alpha S$ processes only the moments of order $p < \alpha$ exist. So the estimation methods based on second order statistics of the data cannot be applied. One solution is to use FLOM of the process [5]. The so-called covariations [4] of two random variables are used instead of second order moments in the analysis. The covariation of two jointly $S\alpha S$ real random variables with dispersions γ_x and γ_y are given as:

$$[X,Y]_{\alpha} = \frac{E[XY^{<p-1>}]}{E[|Y|^{p}]}\gamma_y \tag{3}$$

where $\gamma_y = [Y,Y]_{\alpha}$ is the dispersion of random variable Y and $Y^{<p-1>} = |Y|^{p-2}Y$.

3. Frequency Estimation Problem

In the frequency estimation problem the signal model assumed consists of multiple sinusoids

$$s_n = \sum_{k=1}^{K} A_k \sin\{\omega_k n \ + \theta_k\} \tag{4}$$

observed in additive $S\alpha S$ noise

$$x_n = s_n + z_n, \qquad n = 1,\cdots,N. \tag{5}$$

where A_k is the amplitude, ω_k is the angular frequency, and θ_k is the phase of the kth real sinusoid. K is the number of sinusoids and N is the sample size. x_n and z_n are realizations of observation sequence X_n and $S\alpha S$ noise sequence Z_n, respectively.

When the noise samples are independent and identically distributed, the observation sequence can be modeled as a stable AR-process:

$$X_n = a_1 X_{n-1} + \cdots + a_M X_{n-M} + b_0 Z_n. \tag{6}$$

This leads to the Generalized Yule-Walker Equation when X_{n-m} is given as [5]:

$$\begin{aligned} E[X_n|X_{n-m}] &= a_1 E[X_{n-1}|X_{n-m}] + \cdots \\ &\quad + a_M E[X_{n-M}|X_{n-m}], \end{aligned} \tag{7}$$

$$E[X_{n+l}|X_n] = \lambda(l) X_n \tag{8}$$

where $m = 1,\cdots,M$. If $\lambda(l)$ denotes the covariation coefficient of X_{n+l} with X_n, one can find the AR-parameters by solving the following linear set of equations:

$$\mathbf{C}\mathbf{a} = \boldsymbol{\lambda} \tag{9}$$

with

$$\mathbf{C} = \begin{bmatrix} \lambda(0) & \lambda(-1) & \cdots & \lambda(1-M) \\ \lambda(1) & \lambda(0) & \cdots & \lambda(2-M) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda(M-1) & \lambda(M-2) & \cdots & \lambda(0) \end{bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix}, \qquad \boldsymbol{\lambda} = \begin{bmatrix} \lambda(1) \\ \lambda(2) \\ \vdots \\ \lambda(M) \end{bmatrix}.$$

In the frequency estimation of sinusoids given by the Equations 4 and 5 the sinusoidal signal component can be assumed to be a stable AR process of order $2K$. As in the Gaussian additive noise case, the model order M of the AR model for the noisy signal should be selected higher than $2K$ in order to allow sufficient additional subspace dimension for the noise component. Assuming that the signal and the noise components are stable processes with the same characteristic exponent, their covariation can be calculated as follows:

$$\begin{aligned} [x_j, x_k]_{\alpha} &= [s_j + e_j, s_k + e_k]_{\alpha} \\ &= [s_j, s_k]_{\alpha} + [s_j, e_k]_{\alpha} \\ &\quad + [e_j, s_k]_{\alpha} + [e_j, e_k]_{\alpha} \end{aligned} \tag{10}$$

where $j,k = 1,\ldots,N$. Since the signal and additive noise are assumed to be independent, the cross-covariation of noise and signal components with each other is

$$\begin{aligned} [s_j, e_k]_{\alpha} &= 0 \\ [e_j, s_k]_{\alpha} &= 0. \end{aligned} \tag{11}$$

On the other hand the covariations of the signal component and noise component with themselves are found as:

$$[s_j, s_k]_{\alpha} = \lambda(j-k)\gamma_{s_k} \tag{12}$$

$$[e_j, e_k]_{\alpha} = \delta_{j,k}\gamma_{e_k} \tag{13}$$

where $\delta_{j,k}$ is the Kronecker delta.

The covariation matrix for alpha-stable processes has the same meaning as that of the covariance matrix for Gaussian processes. As one performs eigen-decomposition of the covariation matrix, the larger eigenvalues will correspond to signal subspace eigenvectors and the remaining eigenvectors will constitute the noise subspace. So one can perform eigen-analysis on the covariation matrix and then apply a suitable

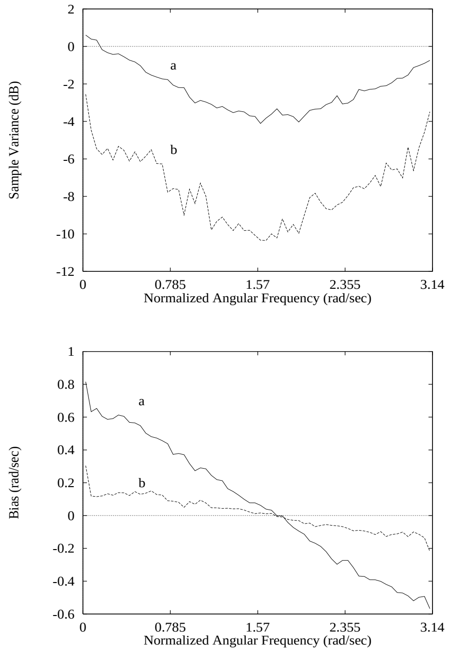

Figure 1. Sample variance and bias of PC-Bartlett and ROC-Bartlett frequency estimators versus normalized angular frequency, a) PC-Bartlett, b) ROC-Bartlett ($\alpha = 1.0$, $p = 0.8$ (ROC-Bartlett), $M = 20$, GSNR $= 5$ dB, $N = 50$, 100 noise realizations, 20 phase realizations).

noise subspace or a signal subspace technique to estimate the parameters. Note that the covariation matrix is not symmetric. This makes the eigen-analysis more complicated and renders many of the subspace-based parameter estimation techniques developed for Gaussian processes unsuitable for the general alpha-stable processes.

One such technique applied to direction of arrival estimation problem is the Robust Covariation-Based MUSIC (ROC-MUSIC) [6]. In this work, we first apply ROC-MUSIC which is a noise subspace method to frequency estimation in alpha-stable environments problem and then we also apply Robust Covariation-Based-Bartlett (ROC-Bartlett) which is a signal subspace method, to the problem.

The second order statistics-based principal component Bartlett frequency estimate is obtained by the peaks of the spectrum estimator [3]:

$$\text{PC-Bartlett}(\omega) = \frac{1}{M} \sum_{i=1}^{2K} \lambda_i \left| \mathbf{d}^H \mathbf{v}_i \right|^2 \qquad (14)$$

where $\mathbf{d}$ is the complex sinusoidal vector $\mathbf{d} = [1 \;\; \exp\{j\omega\} \;\; \cdots \;\; \exp\{j\omega(M-1)\}]$, and λ_i and $\mathbf{v}_i$ are

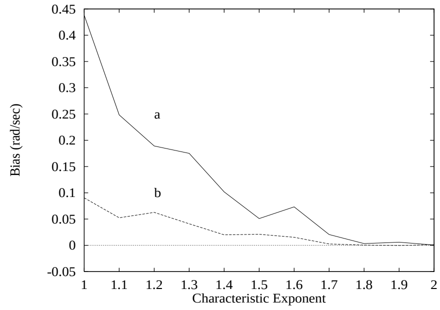

Figure 2. Bias of MUSIC and ROC-MUSIC frequency estimators versus characteristic exponent of alpha-stable noise, a) PC-Bartlett and MUSIC, b) ROC-Bartlett and ROC-MUSIC ($\omega = 0.76$ rad/sec, $M = 20$, GSNR $= 5$ dB, $N = 50$, 100 noise realizations, 20 phase realizations).

ordered eigenvalues such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$, and the corresponding eigenvectors of $M \times M$ autocorrelation matrix. ROC-Bartlett is obtained by substituting the covariation matrix for the autocorrelation matrix.

4. Simulation Experiments

We have used ROC-MUSIC and ROC-Bartlett methods to estimate the frequency of a single real sinusoid. The modified FLOM (MFLOM) estimator given by [6]

$$\hat{\mathbf{C}}(k,l) = \frac{\sum_{i=1}^{N-M+1} X_{k+i-1} |X_{l+i-1}|^{p-2} X_{l+i-1}}{\sum_{i=1}^{N-M+1} |X_{l+i-1}|^{p}}, \quad k,l = 1, \cdots, M, \qquad (15)$$

is defined for moment order $p \in [0, 2]$ and it is used to estimate the (k,l)th element of the sample covariation matrix $\hat{\mathbf{C}}$. M denotes the order of AR-model. We have applied $S\alpha S$ noise sequences with varying α and γ parameters. To generate the $S\alpha S$ noise process we used the method described by Tsihrintzis and Nikias [7] which is a special case of the more general method including the non-symmetric alpha-stable random variable generation given by Chambers, Mallows and Stuck [2]. The moment order p and the sample size N were equal to 0.8 and 50, respectively. The AR-model order was chosen as 20 in the simulations. The *generalized* SNR, GSNR $= 10 \log \left(\frac{1}{\gamma N} \sum_{n=1}^{N} |s(n)|^2 \right)$ is equal to 5 dB.

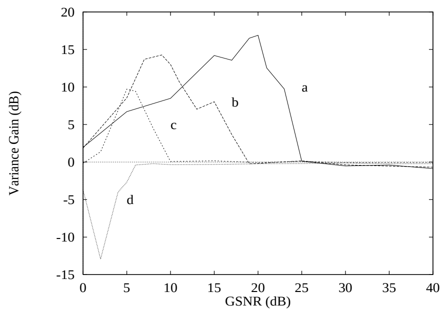

Figure 3. Variance reduction of ROC-Bartlett with respect to PC-Bartlett frequency estimator versus GSNR averaged on the frequency axis, a) $\alpha = 1.0$, b) $\alpha = 1.4$, c) $\alpha = 1.8$, d: $\alpha = 2.0$ ($M = 20$, $N = 50$, 100 noise and phase realizations).

4.1. Frequency Dependence of Bias and Variance

In Figure 1 the sample variance and the bias of PC-Bartlett and ROC-Bartlett frequency estimators are plotted against the angular frequency for $\alpha = 1.0$ (Cauchy distribution) and GSNR = 5 dB. The number of noise realizations and phase realizations are 100 and 20, respectively, making a total of 2000 Monte Carlo runs. The ROC-Bartlett has approximately 5 dB lower sample variance than the PC-Bartlett.

The bias curves depict a symmetry around approximately $\omega = 1.7$ rad/sec. The ROC-Bartlett performs much better than the PC-Bartlett. The difference of their bias value is more than 0.4 rad/sec around $\omega = 0.2$ rad/sec.

4.2. Dependence of Bias upon α

The bias behaviour of the estimators for $\omega = 0.76$ rad/sec as a function of the characteristic exponent α of the noise is shown in Figure 2. The figure indicates that the bias gets smaller as α increases. When $\alpha = 1$ the bias values are 0.45 rad/sec for PC-Bartlett and MUSIC and it is less than 0.1 rad/sec for their ROC versions. As this figure depicts for the single tone case as in our experiments, MUSIC and Bartlett estimators show exactly the same performance.

4.3. Dependence of Variance Reduction upon the GSNR

In Figure 3, the variance reduction achieved by ROC-Bartlett with respect to PC-Bartlett is plotted against GSNR for different values of α. The number of Monte Carlo runs is 100, each with a different noise and phase realization. The curve exhibiting the highest gain belongs to $\alpha = 1.0$ (Cauchy noise). This gain is approximately 17 dB when GSNR = 20 dB. The curves show that the variance increase introduced by the ROC-Bartlett against PC-Bartlett is negligible with the exception of Gaussian noise case where the GSNR threshold of ROC estimator is higher with respect to that of the second order statistics-based estimator. This behaviour validates the robustness of FLOM-based subspace techniques and it is also shared by the noise subspace technique ROC-MUSIC.

5. Conclusion

When the additive noise in the frequency estimation problem can be modeled as an alpha-stable process, the FLOM-based subspace techniques perform better than their second order statistics-based counterparts. Both ROC-MUSIC and ROC-Bartlett methods showed superior performance with respect to MUSIC and PC-Bartlett methods in our experiments, especially for low α values.

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